

Maximum Information and Quantum Prediction Algorithms

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Abstract

This paper describes an algorithm for selecting a consistent set within the consistent histories approach to quantum mechanics and investigates its properties. The algorithm uses a maximum information principle to select from among the consistent sets formed by projections defined by the Schmidt decomposition. The algorithm unconditionally predicts the possible events in closed quantum systems and ascribes probabilities to these events. A simple spin model is described and a complete classification of all exactly consistent sets of histories formed from Schmidt projections in the model is proved. This result is used to show that for this example the algorithm selects a physically realistic set. Other tentative suggestions in the literature for set selection algorithms using ideas from information theory are discussed.

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I. INTRODUCTION

It is hard to find an entirely satisfactory interpretation of the quantum theory of closed systems, since quantum theory does not distinguish physically interesting time-ordered sequences of operators. In this paper, we consider one particular line of attack on this problem: the attempt to select consistent sets by using the Schmidt decomposition together with criteria intrinsic to the consistent histories formalism. For a discussion of why we believe consistent histories to be incomplete without a set selection algorithm see [1,2] and for other ideas for set selection algorithms see [3–6]. This issue is controversial: others believe that the consistent histories approach is complete in itself [7–9].

A. Consistent histories formalism

We use a version of the consistent histories formalism in which the initial conditions are defined by a pure state, the histories are branch-dependent and consistency is defined by Gell-Mann and Hartle’s medium consistency criterion eq. (1.3). We restrict ourselves to closed quantum systems with a Hilbert space in which we fix a split $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$; we write $\dim(\mathcal{H}_j) = d_j$ and we suppose that $d_1 \leq d_2 < \infty$. The model described in sec. IV has a natural choice for the split. Other possibilities are discussed in [3].

Let $|\psi\rangle$ be the initial state of a quantum system. A *branch-dependent set of histories* is a set of products of projection operators indexed by the variables $\alpha = \{\alpha_n, \alpha_{n-1}, \dots, \alpha_1\}$ and corresponding time coordinates $\{t_n, \dots, t_1\}$, where the ranges of the α_k and the projections they define depend on the values of $\alpha_{k-1}, \dots, \alpha_1$, and the histories take the form:

$$C_\alpha = P_{\alpha_n}^n(t_n; \alpha_{n-1}, \dots, \alpha_1) P_{\alpha_{n-1}}^{n-1}(t_{n-1}; \alpha_{n-2}, \dots, \alpha_1) \dots P_{\alpha_1}^1(t_1). \quad (1.1)$$

Here, for fixed values of $\alpha_{k-1}, \dots, \alpha_1$, the $P_{\alpha_k}^k(t_k; \alpha_{k-1}, \dots, \alpha_1)$ define a projective decomposition of the identity indexed by α_k , so that $\sum_{\alpha_k} P_{\alpha_k}^k(t_k; \alpha_{k-1}, \dots, \alpha_1) = 1$ and

$$P_{\alpha_k}^k(t_k; \alpha_{k-1}, \dots, \alpha_1) P_{\alpha'_k}^k(t_k; \alpha_{k-1}, \dots, \alpha_1) = \delta_{\alpha_k \alpha'_k} P_{\alpha_k}^k(t_k; \alpha_{k-1}, \dots, \alpha_1). \quad (1.2)$$

Here and later, though we use the compact notation α to refer to a history, we intend the individual projection operators and their associated times to define the history.

We use the consistency criterion¹

$$D_{\alpha\beta} = 0, \quad \forall \alpha \neq \beta, \quad (1.3)$$

which Gell-Mann and Hartle call *medium consistency*, where $D_{\alpha\beta}$ is the *decoherence matrix*

$$D_{\alpha\beta} = \text{Tr} (C_\alpha \rho C_\beta^\dagger). \quad (1.4)$$

Probabilities for consistent histories are defined by the formula

$$p(\alpha) = D_{\alpha\alpha}. \quad (1.5)$$

With respect to the $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ splitting of the Hilbert space, the *Schmidt decomposition* of $|\psi(t)\rangle$ is an expression of the form

$$|\psi(t)\rangle = \sum_{i=1}^{d_1} [p_i(t)]^{1/2} |w_i(t)\rangle_1 \otimes |w_i(t)\rangle_2, \quad (1.6)$$

where the *Schmidt states* $\{|w_i\rangle_1\}$ and $\{|w_i\rangle_2\}$ form, respectively, an orthonormal basis of \mathcal{H}_1 and part of an orthonormal basis of \mathcal{H}_2 , the functions $p_i(t)$ are real and positive, and we take the positive square root. For fixed time t , any decomposition of the form eq. (1.6) then has the same list of probability weights $\{p_i(t)\}$, and the decomposition (1.6) is unique if these weights are all different. These probability weights are the eigenvalues of the reduced density matrix.

The idea motivating this paper is that the combination of the ideas of the consistent histories formalism and the Schmidt decomposition might allow us to define a mathematically precise and physically interesting description of the quantum theory of a closed system. We consider constructing histories from the projection operators²

¹For a discussion of other consistency criteria see, for example, refs. [10–13].

²There are other ways of constructing projections from the Schmidt decomposition [3], though for the model considered in this paper the choices are equivalent.

$$P_i(t) = |w_i(t)\rangle_1 \langle w_i(t)|_1 \otimes I_2 \text{ and } \overline{P} = I_1 \otimes I_2 - \sum_i P_i(t), \quad (1.7)$$

which we refer to as *Schmidt projections*. If $\dim \mathcal{H}_1 = \dim \mathcal{H}_2$ the complementary projection \overline{P} is zero. In developing the ideas of this paper, we were influenced in particular by Albrecht's investigations [14,15] of the behaviour of the Schmidt decomposition in random Hamiltonian interaction models and the description of these models by consistent histories.

II. INFORMATION

Recent work [3,6,16,17] has shown some of the difficulties in formulating a successful set selection algorithm. The analysis of [3] suggests that in many systems no algorithm that constructs sets by proceeding forwards in time will produce the correct physical set. If so, an algorithm must consider the entire time evolution of a system if it is always to overcome this problem. This paper introduces an algorithm that is global with respect to time: the algorithm considers the class³ of all consistent sets of histories formed from Schmidt projections and selects from among them the one with the greatest Shannon information [18].

Information⁴ is a term often used in the study of quantum mechanics and is used in many different senses. Hartle [17]⁵ considers the *missing information* of a set of histories in a generalised spacetime quantum mechanics — he defines the missing information S of a set of histories \mathcal{S} with initial density matrix ρ as

$$S(\mathcal{S}, \rho) = \max_{\rho' \in \{D(\mathcal{S}, \rho') = D(\mathcal{S}, \rho)\}} E(\rho'), \quad (2.1)$$

where $D(\mathcal{S}, \rho)$ is the decoherence matrix for the set of histories \mathcal{S} with initial density matrix ρ . Throughout this paper E will denote the Shannon information of a set of probabilities

³*Class* is used as a synonym for *set* when referring to a *set* of sets of consistent histories.

⁴*Entropy* or *information-entropy* are used instead by some authors.

⁵For comments on and corrections to Hartle's paper see [19]

or, in the case of a positive definite Hermitian matrix, the Shannon information of its eigenvalues⁶. So, for example, $E(\rho') = -\text{Tr} \rho' \log \rho'$ and

$$E(\mathcal{S}, \rho) = \sum_{\alpha \in \mathcal{S}} -D_{\alpha\alpha} \log D_{\alpha\alpha}, \quad (2.2)$$

where $\{D_{\alpha\alpha}\}$ are the diagonal elements of the decoherence matrix $D(\mathcal{S}, \rho)$. Note that if a set of histories \mathcal{S} is medium consistent then $E(\mathcal{S}, \rho) = E[D(\mathcal{S}, \rho)]$: generically this is not true for weak consistency criteria.

$S(\mathcal{S}, \rho)$ is the information content of a *maximum-entropy* [20] estimation of the initial density matrix given the set of histories and their probabilities—it quantifies what can be inferred about the initial density matrix using the set of histories and their probabilities. Hartle goes on to define

$$S(\mathcal{G}, \rho) = \min_{\mathcal{S} \in \mathcal{G}} S(\mathcal{S}, \rho), \quad (2.3)$$

where \mathcal{G} is some class of consistent sets of histories. Computing $S(\mathcal{G}, \rho)$ for different classes enables one to understand different ways information about a quantum system can be obtained. For example Hartle suggests comparing whether the same information is available using homogeneous [21] histories instead of the more general inhomogeneous histories. When \mathcal{G} is the class of all consistent sets he calls $S(\mathcal{G}, \rho)$ the *complete information*.

Eq. (2.3) could be used as the basis for a set selection algorithm by specifying some class of sets of histories \mathcal{G} and selecting a set of histories that produces the minimum in eq. (2.3). This does not work for general classes, since if the class contains sets of histories which include projections onto the eigenspaces of ρ (in non-relativistic quantum mechanics) these projections completely specify ρ , so a rather uninteresting set of histories is selected. However, if the initial state is pure and a *Schmidt class* (a class of sets of histories formed from Schmidt projections) is used it will not generically contain a set of histories that includes a rank one projection onto the initial state, hence the set of histories selected by

⁶in information theory the singularity for zero probabilities is removed by defining $0 \log 0 = 0$.

eq. (2.3) might not be trivial. For instance the set of histories consisting of projections $P \otimes I$ and $\overline{P} \otimes I$, where P is the projection onto the non-zero system Schmidt eigenspaces, has missing information $\log \text{rank}(P \otimes I)$. It might be considered unnatural to assume a pure initial state and then make a maximum entropy calculation over density matrices of other ranks; however, this idea has a more serious flaw. The aim of set selection algorithms is to make statements concerning physical events, not merely to supply initial conditions. This algorithm only searches for a set of histories that best specifies the initial conditions and there is no reason to expect it to produce sets that do more than describe the initial conditions.

Isham and Linden [6] independently, recently proposed a different version of missing information, which they call *information-entropy*, that is simpler and does not use ideas of maximum entropy.

$$S'(\mathcal{S}, \rho) = - \sum_{\alpha \in \mathcal{S}} D_{\alpha\alpha} \log \frac{D_{\alpha\alpha}}{\widehat{\text{dim}}^2(\alpha)}, \quad (2.4)$$

where

$$\widehat{\text{dim}} \alpha = \frac{\text{Tr}(C_\alpha)}{\text{Tr}(I)} \quad (2.5)$$

is the *normalised dimension* of the history, and C_α and I are considered as operators in the same n -fold tensor product space [22,23] of \mathcal{H} . For example, if the history α is defined by consecutive projections $\{P_k, k = 1, \dots, n\}$ then $\widehat{\text{dim}}(\alpha) = \text{Tr}(P_1 \otimes \dots \otimes P_n)/d^n = \text{rank}(P_1) \times \dots \times \text{rank}(P_n)/d^n$. Like Hartle's missing information, S' *decreases* under refinements and extensions of \mathcal{S} . Isham and Linden show that

$$\min_{\mathcal{S} \in \mathcal{G}} S'(\mathcal{S}, \rho) \geq -\text{Tr} \rho \log \rho - n \log d \quad (2.6)$$

and for some examples that the bound is obtained, and they conjecture that the bound is attained in general. Isham and Linden also suggest that information-entropy might help in the development of a set selection criterion — they suggest that perhaps the minimisation should be carried out with respect to a system–environment split. Clearly some restriction on

the class of sets used is necessary since bound (2.6) contains no mention of the Hamiltonian or time evolution of the system — simply minimising information-entropy is unlikely to produce a good set selection algorithm, since the sets of histories that describes experimental situations are much more than a description of the initial conditions.

Gell-Mann and Hartle discuss similar ideas in detail in ref. [5]. They introduce a measure, which they call *total information* or *augmented entropy*, Σ that combines algorithmic information (see for example ref. [24]), entropy-information and coarse graining. This is an attempt to provide a quantitative measure of *quasiclassicality*. They show that minimising Σ does not provide a useful set selection algorithm — the results are trivial, histories are selected that consist of nothing but projections onto the initial state — but they suggest augmenting the minimisation with a stronger consistency criterion,

$$\langle \alpha | M_\alpha^\dagger M_\beta | \beta \rangle = p_\alpha \delta_{\alpha\beta} \quad \forall \alpha \neq \beta, \quad M_\alpha \in \mathcal{M}_\alpha \text{ and } M_\beta \in \mathcal{M}_\beta, \quad (2.7)$$

where \mathcal{M}_α and \mathcal{M}_β are sets of operators. This is an interesting idea. So far however, Gell-Mann and Hartle have not proposed a definite algorithm for choosing the \mathcal{M}_α . Without a concrete scheme for choosing the sets \mathcal{M}_α the set selection problem of course becomes the problem of selecting \mathcal{M}_α . There seems a risk that Gell-Mann and Hartle's proposal also has the previously mentioned disadvantage of favouring set of histories that only provide a description of the initial state and say nothing about the dynamics, though perhaps with a suitable choice for \mathcal{M}_β this problem would not arise.

The approach we present here starts with a precisely defined class of quasiclassical sets of histories (formed from Schmidt projections) and picks the set of histories from this class with the maximum information.

It might seem counterintuitive to use a maximum information principle, especially as other approaches in the literature to date have looked at minimising measures of information. However, these approaches have started with a much larger class of sets of histories. Picking the set with largest information from these classes would result in a non-quasiclassical set of histories with each history having the same probability. In this approach though, we are

using a highly restricted class — the class formed using Schmidt projections. This class of histories is so restricted that in some cases it may only consist of sets with projections at $t = 0$ onto the initial Schmidt states. Picking the set with the largest information tends to pick the set with the largest number of histories. Other functions of the history probabilities could also be used, the essential requirement being that the functions tend to increase with the number of projections. We regard this proposal as a starting point for further investigations into set selection algorithms — especially since there are only pragmatic rather than fundamental reasons for choosing maximum information as a set selection axiom.

III. ALGORITHM

Let $\mathcal{G}(\mathcal{H}, U, |\psi\rangle)$ be the class of all sets of non-trivial⁷, exactly consistent, branch-dependent⁸ histories formed from Schmidt projection operators, where $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ is a finite Hilbert space, $U(t)$ a time evolution operator and $|\psi\rangle$ the initial state. Note that in this section the set of histories includes the initial state. The algorithm selects the set $\mathcal{S} \in \mathcal{G}$ with the greatest Shannon information. That is

$$\max_{\mathcal{S} \in \mathcal{G}} E(\mathcal{S}) = \max_{\mathcal{S} \in \mathcal{G}} \sum_{\alpha \in \mathcal{S}} -p_{\alpha} \log p_{\alpha}, \quad (3.1)$$

where p_{α} is the probability of history α . The class \mathcal{G} could be chosen differently by using any of the consistency or non-triviality criteria from ref. [3]. Another variant uses sets of histories formed by Schmidt projections onto the system eigenspaces of the individual path-projected-states ($U(t)C_{\alpha}|\psi\rangle$), not the total state, so that the choice of projections is branch-dependent as well as the choice of projection times. This is likely to be necessary in

⁷In this paper we call a history trivial if its probability is zero and non-trivial if its probability is non-zero.

⁸A branch-independent version of the algorithm can be formulated similarly

general to produce realistic sets.

When the initial state is pure, in a Hilbert space of dimension $d (= d_1 d_2)$ there can only be d non-trivial, exactly consistent histories within a set⁹. In realistic examples approximate consistency may have to be considered. To ensure the algorithm is well defined it is important that the number of possible histories is finite, which will only be true if we use a parameterised non-triviality criterion or we use a consistency criterion, such as the DHC, that can only be satisfied by a finite number of histories [13]. This is a natural requirement for any set of histories in a finite Hilbert space since the exactly consistent sets are finite.

To show that the maximum in eq. (3.1) exists we define two sets of histories as *information equivalent*, $\mathcal{S}_1 \sim \mathcal{S}_2$, if $E(\mathcal{S}_1) = E(\mathcal{S}_2)$; that is, sets of histories are information equivalent if they have the same information. Note that information equivalent sets generically are not physically equivalent, but physically equivalent sets are information equivalent. Eq. (3.1) selects an information equivalent class of sets of histories that all have the maximum information. Sufficient conditions for eq. (3.1) to be well defined are that \mathcal{G}/\sim is closed and that $E(\mathcal{S})$ is bounded. \mathcal{G} itself is not closed, but the only limit sets of histories it does not include are those containing zero probability histories, and since zero probability histories contribute zero information these limit sets are equivalent to sets which are in \mathcal{G} , hence \mathcal{G}/\sim is closed. Moreover these limit sets are also physically equivalent to some of the sets that they are information equivalent to, since they only differ by zero probability histories — excluding the limit sets does not change anything physical. The information of any set of histories in \mathcal{G} is bounded, since the number of histories in any set of histories in \mathcal{G} is bounded and the information of a set of n probabilities is bounded by $\log n$. Conditions sufficient to ensure uniqueness are much more complicated. It seems likely that a unique physically equivalent class will generically be selected, but in special cases it is clear that this is not the case.

⁹There can be $2d$ if weak consistency is used.

First we describe some useful properties of this algorithm and then we apply it to a simple model.

A. Completeness

The set of histories selected by the algorithm cannot be extended (except trivially) because any non-trivial extension increases the information content. To see this consider the set of histories \mathcal{S} and an extension \mathcal{S}' . The probabilities for the new histories can be written in the form $p_\alpha q_\beta^{(\alpha)}$ where $\sum_\beta q_\beta^{(\alpha)} = 1$ for all α . The information of the new set is

$$E(\mathcal{S}') = - \sum_\alpha \sum_\beta p_\alpha q_\beta^{(\alpha)} \log p_\alpha q_\beta^{(\alpha)} = E(\mathcal{S}) + \sum_\alpha p_\alpha E(q_\beta^{(\alpha)}), \quad (3.2)$$

which is strictly greater than $E(\mathcal{S})$ whenever the extension results in at least one non-zero probability.

B. Additivity

A set of branch-dependent histories has a probability tree structure, where each history α refers to a terminal node of the tree and the unique path from that node to the root node. The nodes themselves are associated with projection operators and path projected states. Define $\mathcal{S}_{\alpha k}$ to be the set of all histories extending from the k^{th} node of history α , normalised so that the total probability is one. This is a set of histories in its own right which will be consistent if the entire set of histories is consistent. Consider a simple example where the first projection produces two histories with probabilities p and q and the subtrees from these nodes are \mathcal{S}_p and \mathcal{S}_q . The information for the set of histories can then be written,

$$E(\mathcal{S}) = E(\{p, q\}) + pE(\mathcal{S}_p) + qE(\mathcal{S}_q). \quad (3.3)$$

This formula is easy to generalise. Each subtree must have maximum information subject to the constraint that the history vectors span a space orthogonal to the other history states. That is, a global maximum must also be a local maximum in each degree of freedom and the subtrees are the degrees of freedom.

C. Large sets

One of the problems with the algorithms in ref. [3] is their tendency to make projections too early so that they prevent projections at later times. Other problems also arise with algorithms that produce histories with zero or small probabilities. The maximum-information algorithm will not have these problems, since any projection that prevents later extensions is unlikely to be selected, histories with zero probability will never be selected (since they contribute no information), and histories with small probabilities are also unlikely to be selected. Therefore the algorithm is likely to produce large complicated sets of histories.

D. Stability

It is difficult to prove any general results about stability for this algorithm, but it seems likely to produce stable predictions for the following reason. The Schmidt projections and hence decoherence matrix elements generically will vary continuously with sufficiently small changes in the initial state and Hamiltonian, thus the algorithm can be regarded as a continuous optimisation problem, and the solutions to continuous optimisation problems are stable.

IV. A SIMPLE SPIN MODEL

We now consider a simple model in which a single spin half particle, the system, moves past a line of spin half particles, the environment, and interacts with each in turn. This can be understood as modelling either a series of measurement interactions in the laboratory or a particle propagating through space and interacting with its environment. In the first case the environment spin half particles represent pointers for a series of measuring devices, and in the second they could represent, for example, incoming photons interacting with the particle.

Either way, the model omits features that would generally be important. For example, the interactions describe idealised sharp measurements — at best a good approximation to real measurement interactions, which are always imperfect. The environment is represented initially by the product of N particle states, which are initially unentangled either with the system or each other. The only interactions subsequently considered are between the system and the environment particles, and these interactions each take place in finite time. We assume too that the interactions are distinct: the k^{th} is complete before the $(k+1)^{\text{th}}$ begins.

A. Definition of the model

We use a vector notation for the system states, so that if \mathbf{u} is a unit vector in R^3 the eigenstates of $\sigma \cdot \mathbf{u}$ are represented by $|\pm \mathbf{u}\rangle$. With the pointer state analogy in mind, we use the basis $\{|\uparrow\rangle_k, |\downarrow\rangle_k\}$ to represent the k^{th} environment particle state, together with the linear combinations $|\pm\rangle_k = (|\uparrow\rangle_k \pm i|\downarrow\rangle_k)/\sqrt{2}$. We compactify the notation by writing environment states as single kets, so that for example $|\uparrow\rangle_1 \otimes \dots \otimes |\uparrow\rangle_n$ is written as $|\uparrow_1 \dots \uparrow_n\rangle$, and we take the initial state $|\psi(0)\rangle$ to be $|\mathbf{v}\rangle \otimes |\uparrow_1 \dots \uparrow_n\rangle$.

The interaction between the system and the k^{th} environment particle is chosen so that it corresponds to a measurement of the system spin along the \mathbf{u}_k direction, so that the states evolve as follows:

$$|\mathbf{u}_k\rangle \otimes |\uparrow\rangle_k \rightarrow |\mathbf{u}_k\rangle \otimes |\uparrow\rangle_k, \quad (4.1)$$

$$|-\mathbf{u}_k\rangle \otimes |\uparrow\rangle_k \rightarrow |-\mathbf{u}_k\rangle \otimes |\downarrow\rangle_k. \quad (4.2)$$

A simple unitary operator that generates this evolution is

$$U_k(t) = P(\mathbf{u}_k) \otimes I_k + P(-\mathbf{u}_k) \otimes e^{-i\theta_k(t)F_k}, \quad (4.3)$$

where $P(\mathbf{x}) = |\mathbf{x}\rangle\langle\mathbf{x}|$ and $F_k = i|\downarrow\rangle_k\langle\uparrow|_k - i|\uparrow\rangle_k\langle\downarrow|_k$. Here $\theta_k(t)$ is a function defined for each particle k , which varies from 0 to $\pi/2$ and represents how far the interaction has progressed. We define $P_k(\pm) = |\pm\rangle_k\langle\pm|_k$, so that $F_k = P_k(+) - P_k(-)$.

The Hamiltonian for this interaction is thus

$$H_k(t) = i\dot{U}_k(t)U_k^\dagger(t) = \dot{\theta}_k(t)P(-\mathbf{u}_k) \otimes F_k, \quad (4.4)$$

in both the Schrödinger and Heisenberg pictures. We write the extension of U_k to the total Hilbert space as

$$V_k = P(\mathbf{u}_k) \otimes I_1 \otimes \cdots \otimes I_n + P(-\mathbf{u}_k) \otimes I_1 \otimes \cdots \otimes I_{k-1} \otimes e^{-i\theta_k(t)F_k} \otimes I_{k+1} \otimes \cdots \otimes I_n. \quad (4.5)$$

We take the system particle to interact initially with particle 1 and then with consecutively numbered ones, and there is no interaction between environment particles, so that the evolution operator for the complete system is

$$U(t) = V_n(t) \cdots V_1(t), \quad (4.6)$$

with each factor affecting only the Hilbert spaces of the system and one of the environment spins.

We suppose, finally, that the interactions take place in disjoint time intervals and that the first interaction begins at $t = 0$, so that the total Hamiltonian is simply

$$H(t) = \sum_{k=1}^n H_k(t), \quad (4.7)$$

and we have that $\theta_1(t) > 0$ for $t > 0$ and that, if $\theta_k(t) \in (0, \pi/2)$, then $\theta_i(t) = \pi/2$ for all $i < k$ and $\theta_i(t) = 0$ for all $i > k$.

V. CLASSIFICATION OF SCHMIDT PROJECTION CONSISTENT SETS IN THE MODEL

For generic choices of the spin measurement directions, in which no adjacent pair of the vectors $\{\mathbf{v}, \mathbf{u}_1, \dots, \mathbf{u}_n\}$ is parallel or orthogonal, the exactly consistent branch-dependent sets defined by the Schmidt projections onto the system space can be completely classified in this model. The following classification theorem is proved in this section:

Theorem In the spin model defined above, suppose that no adjacent pair of the vectors $\{\mathbf{v}, \mathbf{u}_1, \dots, \mathbf{u}_n\}$ is parallel or orthogonal. Then the histories of the branch-dependent consistent sets defined by Schmidt projections take one of the following forms:

- (i) a series of Schmidt projections made at times between the interactions — i.e. at times t such that $\theta_k(t) = 0$ or $\pi/2$ for all k .
- (ii) a series as in (i), made at times t_1, \dots, t_n , together with one Schmidt projection made at any time t during the interaction immediately preceding the last projection time t_n .
- (iii) a series as in (i), together with one Schmidt projection made at any time t during an interaction taking place after t_n .

Conversely, any branch-dependent set, each of whose histories takes one of the forms (i)-(iii), is consistent.

We assume below that the set of spin measurement directions satisfies the condition of the theorem: since this can be ensured by an arbitrarily small perturbation, this seems physically reasonable. The next sections explain, with the aid of this classification, the results of various set selection algorithms applied to the model.

A. Calculating the Schmidt states

Eq. (4.3) can be written

$$U_j(t) = e^{-i\theta_j(t)P(-\mathbf{u}_j)} \otimes P_j(+) + e^{i\theta_j(t)P_j(-\mathbf{u}_j)} \otimes P_j(-). \quad (5.1)$$

Define $x_{+j}(t) = \exp[-i\theta_j(t)P(-\mathbf{u}_j)]$ and $x_{-j}(t) = x_{+j}^\dagger(t)$ so $U_j(t) = x_{+j}(t) \otimes P_j(+) + x_{-j}(t) \otimes P_j(-)$. Let π be a string of n pluses and minuses, $|\pi\rangle$ denote the environment state $|\pi_1\rangle_1 \otimes \dots \otimes |\pi_n\rangle_n$, $P(\pi) = |\pi\rangle\langle\pi|$ and $x_\pi(t) = x_{\pi_n n}(t) \dots x_{\pi_1 1}(t)$. Then

$$U(t) = \sum_{\pi} x_\pi(t) \otimes P(\pi). \quad (5.2)$$

The time evolution of the initial state $|\psi(0)\rangle = |\mathbf{v}\rangle \otimes |\uparrow_1 \dots \uparrow_n\rangle$, the corresponding reduced density matrix and the Schmidt decomposition can now be calculated,

$$|\psi(t)\rangle = \sum_{\pi} x_{\pi}(t) \otimes P(\pi)|\mathbf{v}\rangle \otimes |\uparrow_1 \dots \uparrow_n\rangle = 2^{-n/2} \sum_{\pi} x_{\pi}(t) |\mathbf{v}\rangle \otimes |\pi\rangle, \quad (5.3)$$

since $P(\pi)|\uparrow_1 \dots \uparrow_n\rangle = 2^{-n/2}|\pi\rangle$. The reduced density matrix is

$$\rho_r(t) = \text{Tr}_E[|\psi(t)\rangle\langle\psi(t)|] = 2^{-n} \sum_{\pi} x_{\pi}(t) P(\mathbf{v}) x_{\pi}^{\dagger}(t). \quad (5.4)$$

This can be further simplified by using the homomorphism between $SU(2)$ and $SO(3)$. Define the rotation operators

$$B_{+k}(t) = P(\mathbf{u}_k) + \cos \theta_k(t) \bar{P}(\mathbf{u}_k) - \sin \theta_k(t) \mathbf{u}_k \wedge, \quad (5.5)$$

$B_{-k}(t) = B_{+k}^T(t)$ and $B_{\pi j k}(t) = B_{\pi_k k}(t) \dots B_{\pi_j j}(t)$. $B_{+k}(t)$ corresponds to a rotation of angle $\theta_k(t)$ about \mathbf{u}_k , and $P(\mathbf{u}_k) = \mathbf{u}_k \mathbf{u}_k^T$, a projection operator on R^3 . Note that $P(\mathbf{u}_k)$ is also used to indicate a projection in the system Hilbert space — its meaning should be clear from the context. $B_{\pi_{1n}}(t)$ will usually be simplified to $B_{\pi}(t)$. Then $x_{\pi_{11}}(t) P(\mathbf{v}) x_{\pi_{11}}^{\dagger}(t) = P[B_{\pi_{11}}(t) \mathbf{v}]$. Eq. (5.4) can then be written

$$\rho_r(t) = 2^{-n} \sum_{\pi} P[B_{\pi}(t) \mathbf{v}]. \quad (5.6)$$

Define $A_j(t) = 1/2[B_{+j}(t) + B_{-j}(t)] = P(\mathbf{u}_j) + \cos \theta_j(t) \bar{P}(\mathbf{u}_j)$ and $A_{jk}(t) = A_k(t) \dots A_j(t)$, then $2^{-n} \sum_{\pi} B_{\pi}(t) = A_{1n}(t)$. $A_{1n}(t)$ will usually be written $A(t)$. Since $P[B_{\pi}(t) \mathbf{v}]$ is linear in $B_{\pi}(t)$ the sum in eq. (5.6) can then be done, so

$$\rho_r(t) = \frac{1 + \sigma \cdot A(t) \mathbf{v}}{2}. \quad (5.7)$$

Generically this is not a projection operator since $|A(t) \mathbf{v}|$ may not equal 1. It is convenient however to define $P(\mathbf{y}) = 1/2(1 + \sigma \cdot \mathbf{y})$ for all $\mathbf{y} \in C^3$, and this extended definition will be used throughout the paper. $P(\mathbf{y})$ is a projection operator if and only if \mathbf{y} is a real unit vector. Eq. (5.7) can now be written as $\rho_r(t) = P[A(t) \mathbf{v}]$.

The eigenvalues of eq. (5.7) are $1/2[1 \pm N(t)]$ and the corresponding eigenstates, for $N(t) \neq 0$, are $|\pm \mathbf{w}(t)\rangle$, where $N(t) = |A(t) \mathbf{v}|$ and $\mathbf{w}(t) = A(t) \mathbf{v} N^{-1}(t)$.

Lemma 1. *Sufficient conditions that $N(t) \neq 0$ for all t are that $\theta_i(t) \geq \theta_j(t)$ for all $i < j$ and $\mathbf{u}_i \cdot \mathbf{u}_{i+1} \neq 0$ for all $i \geq 0$.*

Proof. Suppose $\exists t$ s.t. $N(t) = 0$, $\Rightarrow \det A(t) = 0$, $\Rightarrow \exists j$ s.t. $\det A_j(t) = 0$, $\Rightarrow \theta_j(t) = \pi/2$. Let j be the largest j s.t. $\theta_j(t) = \pi/2$, then $A_i(t) = P(\mathbf{u}_i) \forall i \leq j$ and $\det A_i(t) \neq 0 \forall i > j$, $\Rightarrow N(t) = \|A_{(j+1)n}(t)\mathbf{u}_j\| \prod_{j>i\geq 0} |\mathbf{u}_i \cdot \mathbf{u}_{i+1}|$ and $\det A_{(j+1)n}(t) \neq 0$, $\Rightarrow \exists i$ s.t. $|\mathbf{u}_i \cdot \mathbf{u}_{i+1}| = 0$ #

For the rest of this paper it will be assumed that $\{\theta_i\}$ and $\{\mathbf{u}_i\}$ satisfy the conditions of lemma 1. The condition on the $\{\theta_i\}$ holds so long as the environment spin particles are further apart than the range of their individual interactions. The condition on $\{\mathbf{u}_i\}$ holds generically and is physically reasonable since any realistic experiment will not have exact alignment.

B. Decoherence matrix elements

The Heisenberg picture Schmidt projection operators are

$$P_H^\pm(t) = U^\dagger(t) P[\pm \mathbf{w}(t)] \otimes I_E U(t). \quad (5.8)$$

Eq. (5.8) can be rewritten using eq. (5.2)

$$P_H^\pm(t) = \sum_{\pi} x_{\pi}^\dagger(t) P[\pm \mathbf{w}(t)] x_{\pi}(t) \otimes P(\pi) = \sum_{\pi} P[\pm \mathbf{w}_{\pi}(t)] \otimes P(\pi), \quad (5.9)$$

where $\mathbf{w}_{\pi}(t) = B_{\pi}^T(t) \mathbf{w}(t)$.

Consider the probability of a history consisting of projections at time t and then s , where the projectors are Schmidt projectors.

$$p(\pm\pm) = \|P_H^\pm(s) P_H^\pm(t) |\psi(0)\rangle\|^2. \quad (5.10)$$

Eq. (5.10) simplifies using eq. (5.9) and $P(\pi) |\psi(0)\rangle = 2^{-1/2} |\mathbf{v}\rangle \otimes |\pi\rangle$ to become

$$\begin{aligned} p(\pm\pm) &= \sum_{\pi} \|P[\pm \mathbf{w}_{\pi}(s)] P[\pm \mathbf{w}_{\pi}(t)] |\mathbf{v}\rangle \otimes P(\pi) | \uparrow_1 \dots \uparrow_n \rangle\|^2 \\ &= 2^{-n-2} \sum_{\pi} [1 \pm \mathbf{w}_{\pi}(t) \cdot \mathbf{v}] [1 \pm \mathbf{w}_{\pi}(t) \cdot \mathbf{w}_{\pi}(s)]. \end{aligned} \quad (5.11)$$

The off-diagonal decoherence matrix elements can be calculated similarly.

$$\begin{aligned}
& \langle \psi(0) | P_H^\pm(t) P_H^\pm(s) P_H^\mp(t) | \psi(0) \rangle \\
&= 2^{-n} \sum_{\pi} \text{Tr} \{ P(\mathbf{v}) P[\pm \mathbf{w}_\pi(t)] P[\pm \mathbf{w}_\pi(s)] P[\mp \mathbf{w}_\pi(t)] \} \\
&= 2^{-n-2} \sum_{\pi} [\mathbf{w}_\pi(t) \wedge \mathbf{v}] \cdot [\pm \mathbf{w}_\pi(t) \wedge \mathbf{w}_\pi(s) \pm i \mathbf{w}_\pi(s)]. \tag{5.12}
\end{aligned}$$

For a general set of vectors $\{\mathbf{u}_k\}$ and time functions $\{\theta_k\}$ eqs. (5.11) and (5.12) are very complicated. However, with a restricted set of time functions a complete analysis is possible. The functions $\{\theta_k\}$ are said to describe a *separated interaction* if, for all t , there exists k s.t. $\theta_j(t) = \pi/2$ for all $j < k$, and $\theta_j(t) = 0$ for all $j > k$. For separated interactions a projection time t is said to be *between* interactions j and $j+1$ when $\theta_i(t) = \pi/2$ for all $i \leq j$ and $\theta_i(t) = 0$ for all $i > j$. A projection time t is said to be *during* interaction j when $\theta_i(t) = \pi/2$ for all $i < j$, $\theta_i(t) = 0$ for all $i > j$ and $0 < \theta_j(t) < \pi/2$. Separated interactions have a simple physical meaning: the interactions with the environment spins occur distinctly, and in sequence.

Under this restriction a complete classification of all the consistent sets, both branch dependent and branch independent, is possible. This classification has a particularly simple form for generic \mathbf{v} and $\{\mathbf{u}_k\}$ satisfying $\mathbf{u}_k \cdot \mathbf{u}_{k+1} \neq 0$, and $\mathbf{u}_k \wedge \mathbf{u}_{k+1} \neq 0$ for all $k = 0, \dots, n-1$. Recall $\mathbf{u}_0 = \mathbf{v}$. For weak consistency the second requirement is stronger $(\mathbf{u}_k \wedge \mathbf{u}_{k+1}) \cdot (\mathbf{u}_{k+2} \wedge \mathbf{u}_{k+1}) = \mathbf{u}_k \overline{P}(\mathbf{u}_{k+1}) \mathbf{u}_{k+1} \neq 0$. These assumptions will be assumed to hold unless stated otherwise.

C. Classification theorem

The proof first considers projections at two times and shows that a pair of times gives rise to non-trivial consistent histories only when the earlier time is between interactions or the earlier time is during an interaction and the later time between this interaction and the next. The second part of the proof shows that any set of branch-independent histories consisting of branches that satisfy this rule for all pairs of projections is consistent. The proof holds for weak and medium consistency criteria.

1. Allowed histories

Let t be a time during interaction j . Define $\omega = \theta_j(t)$ and $\phi = \theta_j(s)$. Define $\mathbf{x} = A_{1(j-1)}(s)\mathbf{v} = A_{1(j-1)}(t)\mathbf{v}$ and $\mathbf{y} = A_{(j+1)n}^T(s)A_{1n}(s)\mathbf{v}$. Note $B_{\pi 1n}(t) = B_{\pi 1j}(t)$ and $B_{\pi 1(j-1)}(t) = B_{\pi 1(j-1)}(s)$ since $t < s$. With this notation and using simple vector identities the off-diagonal elements of the decoherence matrix (from eq. 5.12) are

$$2^{-(n+2)} \sum_{\pi} [\mathbf{w}(t) \wedge B_{\pi}(t)\mathbf{v}] \cdot [\pm \mathbf{w}(t) \wedge B_{\pi}(t)\mathbf{w}_{\pi}(s) \pm iB_{\pi}(t)\mathbf{w}_{\pi}(s)]. \quad (5.13)$$

Now

$$B_{\pi}(t)\mathbf{w}_{\pi}(s) = B_{\pi j}(t)B_{\pi 1(j-1)}(t)B_{\pi 1(j-1)}^T(s)B_{\pi jn}^T(s)\mathbf{w}(s) = B_{\pi j}(t)B_{\pi jn}^T(s)\mathbf{w}(s), \quad (5.14)$$

which only depends on π_i for $i \geq j$. Since $B_{\pi 1j}(t)\mathbf{v}$ only depends on π_i for $i \leq j$ the sum eq. (5.14) can be done over all π_i , $i \neq j$.

$$2^{1-j} \sum_{\pi_i, i < j} B_{\pi 1j}(t)\mathbf{v} = [A_j(t) - \pi_j \sin \omega \mathbf{u}_j \wedge] A_{1(j-1)}(t)\mathbf{v} \quad (5.15)$$

$$= \mathbf{w}(t)N(t) - \pi_j \sin \omega \mathbf{u}_j \wedge \mathbf{x}, \quad (5.16)$$

$$2^{-(n-j)} \sum_{\pi_i, i > j} B_{\pi j}(t)B_{\pi jn}^T(s)\mathbf{w}(s) = N^{-1}(s)B_{\pi j}(t)B_{\pi j}^T(s)A_{(j+1)n}^T(s)A_{1n}(s)\mathbf{v} \quad (5.17)$$

$$= N^{-1}(s)B_{\pi j}(t)B_{\pi j}^T(s)\mathbf{y}. \quad (5.18)$$

Substitute these last two results into eq. (5.13) which becomes

$$2^{-3}N^{-1}(s) \sum_{\pi_j} \{ \mathbf{w}(t) \wedge [\mathbf{w}(t)N(t) - \pi_j \sin \omega \mathbf{u}_j \wedge \mathbf{x}] \} \cdot [\pm \mathbf{w}(t) \wedge B_{\pi j}(t)B_{\pi j}^T(s)\mathbf{y} \pm iB_{\pi j}(t)B_{\pi j}^T(s)\mathbf{y}]. \quad (5.19)$$

This can easily be simplified since $\mathbf{w}(t) \wedge \mathbf{w}(t) = 0$. The only remaining term in the first bracket is then linear in π_j , so when the sum over π_j is taken only the terms linear in π_j in the second bracket remain. Eq. (5.19) is therefore

$$1/4N^{-1}(s) \sin \omega \sin(\omega - \phi) [\mathbf{w}(t) \wedge (\mathbf{u}_j \wedge \mathbf{x})] \cdot [\mathbf{w}(t) \wedge (\mathbf{u}_j \wedge \mathbf{y}) \pm i\mathbf{u}_j \wedge \mathbf{y}]. \quad (5.20)$$

Now $\mathbf{w}(t) = [P(\mathbf{u}_j) + \cos \omega \bar{P}(\mathbf{u}_j)]\mathbf{x}N^{-1}(t)$ so $\mathbf{w}(t) \cdot (\mathbf{x} \wedge \mathbf{u}_j) = 0$. Therefore

$$[\mathbf{w}(t) \wedge (\mathbf{u}_j \wedge \mathbf{x})] \cdot [\mathbf{w}(t) \wedge (\mathbf{u}_j \wedge \mathbf{y})] = \mathbf{x}^T \bar{P}(\mathbf{u}_j) \mathbf{y}. \quad (5.21)$$

Also $\mathbf{u}_j \cdot \mathbf{w}(t) = \mathbf{u}_j \cdot \mathbf{x} N^{-1}(t)$ so

$$[\mathbf{w}(t) \wedge (\mathbf{u}_j \wedge \mathbf{x})] \cdot (\mathbf{u}_j \wedge \mathbf{y}) = -N^{-1}(t)(\mathbf{u}_j \cdot \mathbf{x})\mathbf{x} \cdot (\mathbf{u}_j \wedge \mathbf{y}). \quad (5.22)$$

Eq. (5.19) can be simplified using eq. (5.21) and eq. (5.22) to

$$1/4 N^{-1}(s) \sin \omega \sin(\phi - \omega) \{ \pm \mathbf{x}^T \bar{P}(\mathbf{u}_j) \mathbf{y} \pm i N^{-1}(t) (\mathbf{u}_j \cdot \mathbf{x}) \mathbf{u}_j \cdot (\mathbf{x} \wedge \mathbf{y}) \} \quad (5.23)$$

The probabilities can be calculated during the same results. Summing all the terms $i \neq j$ in eq. (5.11) results in

$$\begin{aligned} & 2^{-3} \sum_{\pi_j} \{ 1 \pm \mathbf{w}(t) \cdot [\mathbf{w}(t) N(t) - \pi_j \sin \omega \mathbf{u}_j \wedge \mathbf{x}] \} \left\{ 1 \pm \frac{\mathbf{x}^T A_j(\omega) B_{\pi_j}(t) B_{\pi_j}^T(s) \mathbf{y}}{N(s) N(t)} \right\} \\ &= 2^{-2} [1 \pm N(t)] \left\{ 1 \pm \frac{\mathbf{x}^T [P(\mathbf{u}_j) + \cos \omega \cos(\phi - \omega) \bar{P}(\mathbf{u}_j)] \mathbf{y}}{N(s) N(t)} \right\} \end{aligned} \quad (5.24)$$

$N^2(s) = |A_{1n}(s) \mathbf{v}| = \mathbf{x}^T A_j(\phi) \mathbf{y}$ and $\cos(\omega - \phi) \cos \omega - \cos \phi = \sin \omega \sin(\phi - \omega)$, so eq. (5.24) is

$$1/4 [1 \pm N(t)] \left[1 \pm \frac{N^2(s) + \sin \omega \sin(\phi - \omega) \mathbf{x}^T \bar{P}(\mathbf{u}_j) \mathbf{y}}{N(s) N(t)} \right] \quad (5.25)$$

To write the decoherence matrix without using \mathbf{x} and \mathbf{y} it is necessary to consider three cases: when times s and t are during the same interaction, when they are during adjacent interactions and when they are during separated interactions. If t is during interaction j and s during interaction k the three cases are $k = j$, $k = j + 1$ and $k > j + 1$. For the remainder of this section let $\phi = \theta_k(s)$,

$$N_j(\omega) = |A_j(t) \mathbf{u}_{j-1}| \text{ and } \lambda_{ij} = \prod_{j > k \geq i} |\mathbf{u}_k \cdot \mathbf{u}_{k+1}|. \quad (5.26)$$

Then

$$\mathbf{x} = \lambda_{0(j-1)} \mathbf{u}_{j-1} \quad (5.27)$$

$$N(t) = \lambda_{0(j-1)} N_j(\omega) \quad (5.28)$$

$$N(s) = \lambda_{0(k-1)} N_k(\phi) \quad (5.29)$$

$$\mathbf{y} = \begin{cases} \lambda_{0(j-1)} A_j(s) \mathbf{u}_{j-1} & \text{for } k = j \\ \lambda_{0j} A_{j+1}^2(s) \mathbf{u}_j & \text{for } k = j + 1 \\ \lambda_{(j+1)(k-1)} \lambda_{0(k-1)} N_k^2(\phi) \mathbf{u}_{j+1} & \text{for } k > j + 1 \end{cases} \quad (5.30)$$

The probabilities of the histories (eq. 5.24) are

$$p(\pm\pm) = 1/4[1 \pm \lambda_{0(j-1)} N_j(\omega)][1 \pm a] \quad (5.31)$$

where

$$a = \begin{cases} \frac{N_j^2(\phi) + \sin \omega \cos \phi \sin(\phi - \omega) |\mathbf{u}_{j-1} \wedge \mathbf{u}_j|^2}{N_j(\omega) N_j(\phi)} & \text{for } k = j \\ \frac{\lambda_{(j-1)j} N_{j+1}^2(\phi) + \cos \omega \sin \omega \lambda_{j(j+1)}^2 \sin^2 \phi \mathbf{u}_{j-1}^T \bar{\mathbf{P}}(\mathbf{u}_j) \mathbf{u}_{j+1}}{N_j(\omega) N_{j+1}(\phi)} & \text{for } k = j + 1 \\ N_k(\phi) \frac{\lambda_{(j-1)(k-1)} + \lambda_{(j+1)(k-1)} \cos \omega \sin \omega \mathbf{u}_{j-1}^T \bar{\mathbf{P}}(\mathbf{u}_j) \mathbf{u}_{j+1}}{N_j(\omega)} & \text{for } k > j + 1 \end{cases} \quad (5.32)$$

The nonzero off-diagonal terms are (eq. 5.23)

$$\begin{cases} \frac{\lambda_{0(j-1)} \sin \omega \sin(\phi - \omega) \cos \phi |\mathbf{u}_{j-1} \wedge \mathbf{u}_j|^2}{4 N_j(\phi)} & \text{for } k = j \\ \frac{\lambda_{0(j-1)} \lambda_{j(j+1)} \sin \omega \cos \omega \sin^2 \phi [N_j(\omega) \mathbf{u}_{j-1}^T \bar{\mathbf{P}}(\mathbf{u}_j) \mathbf{u}_{j+1} \pm i \lambda_{(j-1)j} \mathbf{u}_{j-1} \cdot (\mathbf{u}_j \wedge \mathbf{u}_{j+1})]}{4 N_j(\omega) N_{j+1}(\phi)} & \text{for } k = j + 1 \\ \frac{\lambda_{0(j-1)} \lambda_{(j+1)(k-1)} N_k(\phi) \sin \omega \cos \omega [N_j(\omega) \mathbf{u}_{j-1}^T \bar{\mathbf{P}}(\mathbf{u}_j) \mathbf{u}_{j+1} \pm i \lambda_{(j-1)j} \mathbf{u}_{j-1} \cdot (\mathbf{u}_j \wedge \mathbf{u}_{j+1})]}{4 N_j(\omega)} & \text{for } k > j + 1. \end{cases} \quad (5.33)$$

The off-diagonal terms can be zero for two reasons, either there is a degeneracy in the measurement spin directions, or s and t take special values. The necessary and sufficient conditions for the measurement spin directions not to be degenerate is that for all j $\mathbf{u}_j \cdot \mathbf{u}_{j+1} \neq 0$ and $\mathbf{u}_j \wedge \mathbf{u}_{j+1} \neq 0$. The first condition ensures that $\lambda_{ij} \neq 0$ for all i and j and that the Schmidt states are well defined. These cases do not need to be considered when we are interested in exact consistency because they have measure zero and *almost surely* under any perturbation the degeneracy will be lifted. If weak consistency is used only the real part needs to vanish and the measurement direction need to satisfy the stronger condition $\mathbf{u}_{j-1}^T \bar{\mathbf{P}}(\mathbf{u}_j) \mathbf{u}_{j+1} \neq 0$ for all j . This is still of measure zero. If approximate consistency is being

considered the situation is more complicated as the histories will remain approximately consistent under small enough perturbations. This will not be considered in this letter. Unless said otherwise it will be assumed that the measurement spin direction are not degenerate.

Therefore from eqs. (5.33) the only pairs of times giving rise to consistent projections are repeated projections (that is $s = t$ which implies $j = k$ and $\omega = \phi$), projections in between interactions and any later time (that is $\omega = 0$ or $\pi/2$), and a projection during an interaction and a projection at the end of the same interaction (that is $j = k$ $\omega \in [0, \pi/2]$ and $\phi = \pi/2$.)

2. Probabilities of allowed histories

The model is invariant under strictly monotonic reparameterisations of time, $t \rightarrow f(t)$. Therefore for separated interactions no generality is lost by choosing the time functions $\{\theta_j\}$ such that the j^{th} interaction finishes at $t = j$, that is $\theta_i(j) = \pi/2$ for all $i \leq j$ and $\theta_i(j) = 0$ for all $i > j$. It is convenient to define $R_{\pi ij} = [P(\mathbf{u}_i) - \pi_i \mathbf{u}_i \wedge] \dots [P(\mathbf{u}_i) - \pi_i \mathbf{u}_i \wedge]$. Then $B_\pi(m) = R_{\pi 1m}$.

Consider the history α that consists of projections at times $\{m_i : i = 1, 2, \dots, l\}$, then at time $t \in (k-1, k)$ and then at time k , where $\{m_i, k\}$ is an ordered set of positive integers. This history means that the particle spin was in direction $\pm \mathbf{u}_{m_i}$ at time m_i , $i = 1, \dots, l$, direction $\pm \mathbf{w}(t)$ at time t and direction $\pm \mathbf{u}_k$ at time k . Define $\mathbf{u}_0 = \mathbf{v}$ and $m_0 = 0$.

Using the same method as for two projections the probability for history α is

$$p_\alpha = 2^{-n} 2^{-(l+2)} \sum_{\pi} \prod_{i=0}^{l-1} [1 + \alpha_i \alpha_{i+1} \mathbf{w}_\pi(m_i) \cdot \mathbf{w}_\pi(m_{i+1})] \\ \times [1 + \alpha_l \alpha_t \mathbf{w}_\pi(m_l) \cdot \mathbf{w}_\pi(t)] \times [1 + \alpha_t \alpha_k \mathbf{w}_\pi(t) \cdot \mathbf{w}_\pi(m_k)] \quad (5.34)$$

Now

$$\mathbf{w}_\pi(m_i) \cdot \mathbf{w}_\pi(m_{i+1}) = \mathbf{u}_{m_i}^T R_{\pi 1m_i} R_{\pi 1m_{i+1}}^T \mathbf{u}_{m_{i+1}} = \mathbf{u}_{m_i}^T R_{\pi(m_i+1)m_{i+1}} \mathbf{u}_{m_{i+1}}, \quad (5.35)$$

which only depends on π_j for $m_{i+1} \geq j > m_i$. Also

$$\mathbf{w}_\pi(t) \cdot \mathbf{w}_\pi(k) = N_k^{-1}(t) \mathbf{u}_{k-1}^T A_k(t) B_{k\pi_k}(t) \mathbf{u}_k = N_k^{-1}(t) (\mathbf{u}_{k-1} \cdot \mathbf{u}_k), \quad (5.36)$$

which is independent of π and

$$\mathbf{w}_\pi(t) \cdot \mathbf{w}_\pi(m_l) = N_k^{-1}(t) \mathbf{u}_{k-1}^T A_k(t) B_{\pi_k k}(t) R_{\pi(m_l+1)(k-1)} \mathbf{u}_{m_l}, \quad (5.37)$$

which only depends on π_j for $j > m_l$. These last three equations show that each $B_{\pi_i i}$ is linear so the sum over π is trivial and each $B_{\pi_i i}$ can be replaced by A_i .

$$2^{m_i - m_{i+1} - 1} \sum_{\pi_j, m_{i+1} > j > m_i} \mathbf{w}_\pi(m_i) \cdot \mathbf{w}_\pi(m_{i+1}) = \mathbf{u}_{m_i}^T P(\mathbf{u}_{m_{i+1}}) \cdots P(\mathbf{u}_{m_{i+1}-1}) \mathbf{u}_{m_{i+1}} = \lambda_{m_i m_{i+1}}, \quad (5.38)$$

$$2^{m_l - k} \sum_{\pi_i, k \geq i > m_l} \mathbf{w}_\pi(t) \cdot \mathbf{w}_\pi(m_l) = N_k^{-1}(t) \mathbf{u}_{k-1}^T A_k^2(t) \mathbf{u}_{k-1} \lambda_{m_l(k-1)} = \lambda_{m_l(k-1)} N_k(t) \quad (5.39)$$

Using these results to do the sum over all π eq. (5.34) is

$$p_\alpha = 2^{-(l+2)} [1 + \alpha_l \alpha_t \lambda_{m_l(k-1)} N_k(t)] [1 + \alpha_t \alpha_k N_k^{-1}(t) (\mathbf{u}_{k-1} \cdot \mathbf{u}_k)] \prod_{i=0}^{l-1} [1 + \alpha_i \alpha_{i+1} \lambda_{m_i m_{i+1}}]. \quad (5.40)$$

3. Consistency of allowed histories

Since a coarse graining of a consistent set is consistent it is sufficient to only consider the off-diagonal decoherence matrix elements between the most finely grained allowed histories, which are those that consist of projections between all interactions and one projection during the interaction before the final projection. The off-diagonal elements of the decoherence matrix arise from only three forms, which depend on where the two branches separate, that is the earliest projector where they differ.

First consider the case where two histories differ at a projection in between interactions and all projections up to that point have also been in between interactions. Let $C_\alpha = Q_\alpha P_H(k) \cdots P_H(1)$ and $C_\beta = Q_\beta \overline{P}_H(k) \cdots P_H(1)$. The decoherence matrix element between them is

$$2^{-n} \sum_{\pi} \text{Tr} \{ Q_\pi P(\mathbf{u}_k) x_\pi(k) P[\mathbf{w}_\pi(k-1)] \cdots P[\mathbf{w}_\pi(1)] P(\mathbf{v}) \\ \times P[\mathbf{w}_\pi(1)] \cdots P[\mathbf{w}_\pi(k-1)] x_\pi^\dagger(k) P(-\mathbf{u}_k) \} \quad (5.41)$$

where $Q_\pi = \langle \pi | x_\pi(k) Q_\alpha^\dagger Q_\beta x_\pi^\dagger(k) | \pi \rangle$. Since Q_α and Q_β only contain projections after interaction k has completed Q_π is independent of π_j for all $j \leq k$. Now $P[\mathbf{w}_\pi(j)]P[\mathbf{w}_\pi(j-1)]P[\mathbf{w}_\pi(j)] = 1/2(1 + \mathbf{u}_{j-1} \cdot \mathbf{u}_j)P[\mathbf{w}_\pi(j)]$. Let $\mu = 2^{1-m} \prod_{0 < j < m} (1 + \mathbf{u}_{j-1} \cdot \mathbf{u}_j)$ and eq. (5.41) is

$$\mu 2^{-n} \sum_{\pi} \text{Tr}\{Q_\pi P(\mathbf{u}_k) P[B_\pi(k) \mathbf{w}_\pi(k-1)] P(-\mathbf{u}_k)\} \quad (5.42)$$

But $1/2 \sum_{\pi_k} P[B_\pi(k) \mathbf{w}_\pi(k-1)] = P[\mathbf{u}_k(\mathbf{u}_k \cdot \mathbf{u}_{k-1})]$ and $P(\mathbf{u}_k)P[\mathbf{u}_k(\mathbf{u}_k \cdot \mathbf{u}_{k-1})]P(-\mathbf{u}_k) = 0$ so eq. (5.42) is zero.

Now consider $C_\alpha = P_H(k)P_H(t)P_H(k-1) \dots P_H(1)$ and $C_\beta = P_H(k)\bar{P}_H(t)P_H(k-1) \dots P_H(1)$. The decoherence matrix element between them is

$$\mu 2^{-n} \sum_{\pi} \text{Tr}\{P[\mathbf{w}_\pi(k)]P[\mathbf{w}_\pi(t)]P[\mathbf{w}_\pi(k-1)]P[-\mathbf{w}_\pi(t)]P[\mathbf{w}_\pi(k)]\}, \quad (5.43)$$

which, because $B_{\pi_k k} \mathbf{u}_k = \mathbf{u}_k$ equals

$$\mu 2^{-n} \sum_{\pi} \text{Tr}\{P(\mathbf{u}_k)P[\mathbf{w}(t)]P[B_{\pi_k k}(t) \mathbf{u}_{k-1}]P[-\mathbf{w}(t)]P(\mathbf{w}(k))\}. \quad (5.44)$$

The sum over π_k can be done to give $P[\mathbf{w}(t)]P[A_k(t) \mathbf{u}_{k-1}]P[-\mathbf{w}(t)]$, and since $\mathbf{w}(t)$ is parallel to $A_k(t) \mathbf{u}_{k-1}$, eq. (5.44) is zero.

The final case to consider is when then the histories α and β differ in their final projection. They will be trivially consistent.

VI. THE ALGORITHM APPLIED TO THE SPIN MODEL

A set of histories that maximises information must be complete, therefore all histories must consist of projections at times $\{1, \dots, k-1, t, k : t \in (k-1, k)\}$. First we show that k must be the same for all histories, then we show that generically $k = n$. That is, the algorithm selects a branch independent set that generically describes a measurement at the end of each interaction plus one measurement during the final interaction.

The information content of two subtrees rooted at the same point only depends on the projection times within each one. Either the two subtrees have the same information,

in which case their projection times must be the same, or one has more, but since the projection times used in the subtree with greater information will also be consistent if used in the subtree with less information these projection times can be used instead. Therefore in the set with maximum information all the subtree must have the same projection times, thus all the histories must have the same projection times — the maximal set is branch independent.

Let the projection times be $\{1, \dots, k-1, t, k : t \in (k-1, k)\}$. Then from eq. (5.40) and eq. (3.2) the information content of this set is

$$f[N_k(\theta_k(t))] + f[(\mathbf{u}_k \cdot \mathbf{u}_{k-1})N_k^{-1}(\theta_k(t))] + \sum_{k>j>0} f(\mathbf{u}_{j-1} \cdot \mathbf{u}_j) \quad (6.1)$$

where

$$f(x) = -\frac{1+x}{2} \log \frac{1+x}{2} - \frac{1-x}{2} \log \frac{1-x}{2}. \quad (6.2)$$

Maximising eq. (6.1) with respect to t yields

$$E(\mathcal{S}_k) = E_k = 2f(|\mathbf{u}_k \cdot \mathbf{u}_{k-1}|^{1/2}) + \sum_{k>j>0} f(\mathbf{u}_{j-1} \cdot \mathbf{u}_j), \quad (6.3)$$

where \mathcal{S}_k is the branch independent set consisting of projections at times $\{1, \dots, k-1, t_k, k\}$. This is usually maximised by $k = n$ but depending on the relationships between the \mathbf{u}_j any value of k may be possible. For example, consider $\mathbf{u}_{j-1} \cdot \mathbf{u}_j = 1 - \epsilon$ for all $j \neq k$ and $\mathbf{u}_{k-1} \cdot \mathbf{u}_k = \epsilon$ and ϵ is small.

$$E_m = \begin{cases} O(\epsilon \log \epsilon), & \text{for } m < k, \\ 2 \log 2 + O(\epsilon \log \epsilon), & \text{for } m = k, \\ \log 2 + O(\epsilon \log \epsilon) & \text{for } m > k, \end{cases} \quad (6.4)$$

which for small ϵ is maximised by E_k .

The precise relationship between the $\{\mathbf{u}_j\}$ that ensure $E_n E_k$ for all $k < n$ is complicated in detail, but simple qualitatively. Roughly speaking, $E_n < E_k$ only if $|\mathbf{u}_{j-1} \cdot \mathbf{u}_j| \gg |\mathbf{u}_{k-1} \cdot \mathbf{u}_k|$ for all $j > k$, that is all the measurement directions must be approximately parallel after the k^{th} . Monte Carlo integration over $\{\mathbf{u}_i\}$ (with the $SO(3)$ invariant measure) shows that

for $n = 3$ set \mathcal{S}_n is selected 85.7% of the time, for $n = 4$ it is selected 84.3% of the time, and for all $n > 4$ it is selected 84.2% of the time. When the vectors are approximately parallel, that is $|\mathbf{u}_{j-1} \cdot \mathbf{u}_j| = 1 - O(\epsilon)$, set \mathcal{S}_n is selected with probability $1 - O(\epsilon)$. If however all the measurement spins are approximately parallel ($|\mathbf{u}_{j-1} \cdot \mathbf{u}_j| > 1 - \epsilon$, and $-n\epsilon \log \epsilon < 4 \log 2$) then for some orientations of the initial system spin ($\mathbf{v} = \mathbf{u}_0$) $E_1 > E_k$ for all E_k so set \mathcal{S}_1 is selected. That is, the maximal set consists only of a projection during the first interaction and at the end of the first interaction.

Though the results of the algorithm may seem counterintuitive the following discussion shows why this is not a problem.

First consider the case when the system is genuinely closed. All the projections before the last interaction are natural, in the sense that they agree with our intuitive understanding of a measurement type process. It is only the projections during the last interaction, which occur when the set of histories is nearly complete, that are unnatural. Our intuition about the system and the result we believe to be correct relies on the experiment being embedded in a larger system in which the sets of histories considered are always far from complete.

Second consider the case where the system is approximately closed. Then the sets \mathcal{S}_k should describe the first projections of a maximum-information solution in a larger Hilbert space. For reasons explained below, no non-trivial projections onto the system space will result in consistent extensions of the sets \mathcal{S}_k , even if the system interacts with new degrees of freedom in the environment. This shows that though it is a maximum-information set for a subsystem, it is unlikely to be part of the maximum-information set for the entire system. The set most likely to be part of the maximum-information set is the natural set, the set that consists of projections only at the end of each interaction.

The set of normalised histories (in the Schrödinger picture at time k , that is the path-projected states) is

$$\mathcal{S}_k = \{|\alpha_0 \mathbf{v}_k\rangle \otimes |\alpha_1(\uparrow), \dots, \alpha_{k-1}(\uparrow), \alpha_k(\rightarrow), \uparrow_{k+1}, \dots, \uparrow_n\rangle \forall \alpha \in Z_2^{k+1}\}, \quad (6.5)$$

where α is a string of 2^{k+1} plusses and minuses, $+(\uparrow) = \uparrow$, $-(\uparrow) = \downarrow$ and $\pm(\rightarrow)$ are

orthogonal vectors depending on \mathbf{u}_{k-1} and \mathbf{u}_k . This set of histories cannot be non-trivially extended with Schmidt projections (see sec. V). The reason for this is clear. Consider two of the histories $|\pm\mathbf{v}_k\rangle \otimes |e\rangle$ where $|e\rangle$ is the environment state. These histories are only orthogonal because of the orthogonality of the system part of the states. There can be no future non-trivial extensions unless there is an exact degeneracy, because consistency terms between these two histories will contain terms like $|\langle\mathbf{v}|P(\mathbf{w})|\mathbf{v}\rangle| = \sqrt{1/2(1 + \mathbf{v}\cdot\mathbf{w})}$, which is only zero when $\mathbf{w} = -\mathbf{v}$. In contrast if projections are only made at the end of interactions all the histories are orthogonal in the environment Hilbert space of the finished interactions. Unless these interactions are “undone” these histories will always remain orthogonal and cannot interfere. This argument suggests that the true maximum-information set for the total Hilbert space starts off with projections at the end of every interaction but at *no* interior times.

This suggests that an algorithm designed to produce a maximum-information set for a subsystem could be constructed by requiring that all the histories in a set were orthogonal in the environment space, that is the reduced density matrices in the environment Hilbert space for each history are orthogonal. This is equivalent to considering sets of histories that satisfy the strong consistency criterion (2.7) when the set $\{\mathcal{M}_\alpha\}$ is chosen to be $\{P \otimes I : \text{for all projectors } P \text{ on } \mathcal{H}_1\}$.

VII. OTHER ALGORITHMS

Let $\mathcal{G}(\mathcal{H}, U, |\psi\rangle)$ be the class of all sets of non-trivial, exactly consistent, branch-dependent histories formed from Schmidt projection operators in the spin model. Consider an algorithm that selects the set in \mathcal{G} that minimises Isham and Linden’s information-entropy (2.4). Due to the special symmetries of the spin model the selected set will be branch independent — the argument at the start of section (VI) is valid.

Consider the set of projections at m times, so that the normalised dimension of each history is 2^{-m} . Information-entropy for this set is

$$S' = - \sum_{\alpha \in \mathcal{S}} p_{\alpha} \log \frac{p_{\alpha}}{(1/2)^{2m}} = -2m \log(2) - \sum_{\alpha \in \mathcal{S}} p_{\alpha} \log p_{\alpha} . \quad (7.1)$$

Using the notation of the previous section this can be written

$$S'' = - \sum_{m > k > 0} [2 \log 2 - f(\alpha_k)] , \quad (7.2)$$

where the α_k depend on the projection times and vary between -1 and 1 . Since $f(x) \leq \log 2$ each term in the sum is always negative so the minimum occurs for $m = n + 1$, and the selected set consists of projections at the end of every interaction and a projection either at the end or the beginning of the last interaction — the algorithm has selected a natural set. The minimum-information-entropy algorithm selects a set with as many projections as possible, and among these sets it selects the set whose probabilities have the lowest Shannon information. One drawback with this approach is that unless trivial histories are excluded, or the number of histories in a set bounded, the minimum may not exist and the algorithm would therefore be ill defined. In particular if an infinite number of repeated projections are allowed the algorithm is ill defined.

VIII. CONCLUSIONS

This paper defines a precise algorithm for making probabilistic predictions for closed quantum systems. The algorithm considers the class of all non-trivial, exactly consistent, branch-dependent sets of histories defined by Schmidt projections with respect to a fixed split of the Hilbert space and selects from among them the set with the maximum Shannon information. The algorithm avoids many of the problems of the algorithms considered in ref. [3]. Because it considers the entire time evolution of a system — roughly speaking it is global in time, whereas the algorithms in ref. [3] are local — it does not make unphysical projections in systems where recoherence occurs and it produces complete sets of histories that describe the correlations between the system and the environment. Trivial and very small probability histories, which cause problems for some of the algorithms considered in ref. [3] by preventing later physical projections, are unlikely to be selected since they

contribute little information. The algorithm is also likely to be stable under perturbations in the initial conditions, the Hamiltonian and the parameters, since it involves maximising a continuous function.

Section VI has shown that the algorithm selects a natural set for a simple spin model. It would be interesting to test out the algorithm on more realistic examples; however, it seems difficult to apply the algorithm directly, because of the large size and complicated nature of \mathcal{G} . Analytic calculations are only possible when the system is very simple and in more realistic examples computer simulations will be necessary. However, it should be possible at least to get some insight into the algorithm's predictions by maximising subject to constraints, that is by considering a more computationally tractable subset of \mathcal{G} . For example, we could choose a time interval T that is greater than the time of individual interactions (within the particular system) and larger than any timescale over which recoherence occurs. This would be used as a moving time-window over which to perform the maximisation. The earliest projection within each time-window would be selected and the next time-window would commence from that time. Such algorithms should select the same set as a global algorithm if T is large enough, and are also independently interesting.

Because the algorithm predicts the probabilities for events *and* the set of possible events the algorithm is falsifiable: the algorithm is wrong if it selects any sets that do not agree with our experiences. The algorithm can also be applied to situations where we have no experience of what the natural sets of histories are: for example, a (finite) closed system of electrons and photons — and perhaps ultimately could be applied to theories of quantum cosmology.

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